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Gauge Gravity Vacuum in Constraintless Clairaut-Type Formalism

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Abstract: The gauged Lorentz theory with torsion has been argued to have an effective theory whose non-trivial background is responsible for background gravitational curvature if torsion is treated as a quantum-mechanical variable against a background of constant curvature. We use the CDG decomposition to argue that such a background can be found without including torsion. Adapting our previously published Clairaut-based treatment of QCD, we go on to study the implications for second quantisation.

Keywords: gravity; Clairaut equation; Cho–Duan–Ge decomposition; constraintless formalism

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1. Introduction

The dual superconductor model of QCD confinement requires the vacuum to contain a condensate of (chromo) magnetic monopoles. This led several authors to consider embedded, usually Abelian, subgroups within gauge groups. The early focus was on the $U(1)$ subgroup of $SU(2)$, with analyses by Savvidy [1], Nielsen and Olesen [2] and t'Hooft [3] considering the maximal Abelian gauge in which the Abelian subgroup is assumed to lie along the internal e_3 axis. While they did find a magnetic condensate to be a lower energy state than the perturbative vacuum, their analyses blatantly violated gauge covariance and offered no evidence that the chromomagnetic background was due to monopoles. There was also considerable controversy regarding the stability of such a vacuum. These issues were resolved by the Cho–Duan–Ge (CDG) decomposition [4,5], which introduces an internal vector to covariantly allow a subgroup embedding within a theory's gauge group to vary throughout spacetime. Analyses based on this approach confirmed this magnetic background [1,3] and careful consideration of renormalisation and causality [6–9] finally resolved such a condensate to be stable through several independent arguments.

It is common for analyses of QCD based on the CDG decomposition to assume the monopole condensate comprising the vacuum to provide a slow-moving vacuum background to the quantum degrees of freedom (DOFs) [6]. This was the basis of a novel approach to Einstein–Cartan gravity, in which contorsion (or torsion) is the quantised dynamic degree of freedom confined by a slow-moving classical background gravitational curvature et al. [10–12]. Their work was based on the Lorentz gauge field theory initially put forward by Utiyam, Kibble and Sciama [13–15] for which it has long been known that the non-compact nature of the Lorentz group led to the theory not being positive semi-definite. They dealt with this by performing their initial analyses in Euclidean space, transforming the Lorentz gauge group to $SO(4) \simeq SU(2) \times SU(2)$, until later work found the theory to be well defined with propagators for its canonical DOFs [16].

The theory considered in this paper is also a Lorentz gauge theory quadratic in curvature except that we set contorsion to be zero. Instead of including contorsion, we consider the Abelian decomposition of the Lorentz gauge field, whose details and consequences would be obscured by the complexities of handling contorsion properly.

Because we also deal with the non-compact nature of the Lorentz group by working with $SU(2) \times SU(2)$ in Euclidean space [16], we can draw on a considerable body of literature concerning the Abelian decomposition of $SU(2)$ Yang–Mills theory and find that an interesting structure emerges without the introduction of contorsion. Additionally, like Pak et al. [16], we take our DOFs to be those of the Lorentz gauge fields instead of the metric and/or vierbein. To avoid third-order derivatives from entering the equations of motion (EOMs), our theory does not include localised translation symmetry (for which vierbein are required), despite it being accepted that spacetime respects the full Poincaré symmetry group. We restrict ourselves to the subgroup in this work to avoid complications and so that we can find conventional propagators for the gauge bosons with a Lagrangian quadratic in gravitational curvature. We remain mindful, however, that this is a reduced symmetry group of gravitational dynamics rendering our model to be either low-energy effective or perhaps even just a toy.

One of the more confusing mathematical subtleties of the CDG decomposition was the number of canonical degrees of freedom. Shabanov argued that an additional gauge-fixing condition is needed to remove a supposed "two extra degrees" [17] introduced by the internal unit vector field to covariantly describe the embedded subgroup(s). Bae, Cho and Kimm later clarified that this internal vector did not introduce two degrees of freedom requiring to be fixed but non-canonical DOFs without EOMs [18], while the proposed constraint was merely a consistency condition. The interested reader is referred to [6,19–21] for further details (see, also [22,23]). Cho et al. [24] approached the issue with Dirac quantisation using second-order restraints. In an earlier paper [25], however, the authors present a new approach to rigorously elucidate the dynamic DOFs from the topological. It is based on the Clairaut-type formulation, proposed by one of the authors (SD) [26,27], in a constraintless generalisation of the standard Hamiltonian formalism to include Hessians with zero determinant. It provides a rigorous treatment of the non-physical DOFs in the derivation of EOMs and the quantum commutation relations. In this paper, we apply our Clairaut approach to the gauged Lorentz group [28,29] theory with a Lagrangian quadratic in curvature.

A review of the CDG decomposition is given in Section 2, beginning with an introduction in the context of QCD before illustrating its application to $SU(2) \times SU(2)$. In Section 3, we illustrate the reduction of our theory to two copies of two-colour QCD and use one-loop results from the latter to inform us about the former. Section 4 gives a brief overview of the Clairaut–Hamiltonian formalism and uses it to study the quantisation of this theory, sorting canonical dynamic DOFs from DOFs describing the embedding of important subgroups and finding deviations from canonical second quantisation even for dynamic fields. We consider the one-loop effective dynamics in Section 5, discussing the effective particle spectrum in Section 5.1 and the possible emergence of the Einstein–Hilbert (EH) term in Section 5.2. Our final discussion is in Section 6.

2. A Review of the Covariant Abelian Decomposition of Lorentz Gauge Theory

2.1. The CDG Decomposition in $SU(2)$ QCD

2.1.1. Formalism

Abelian dominance has played a major role in our understanding of the QCD vacuum, facilitating the demonstration of a monopole condensate. That a magnetic condensate suitable for colour confinement can have lower energy than the perturbative vacuum has been known since the 1970s [1–3], but in early work the internal direction supporting the magnetic background could not be specified in a covariant manner and nor was there support for the magnetic condensate being due to monopoles. The apparent existence of destabilising tachyon modes was also an issue for some time [2,8,30]. These issues were rectified by the introduction of the CDG decomposition, which specifies the internal direction of the Abelian subgroup in a gauge covariant manner, allowing the internal direction to vary arbitrarily throughout spacetime.

The application of the CDG decomposition in N-colour ($SU(N)$) QCD is as follows: The Lie group $SU(N)$ has $N^2 - 1$ generators $\lambda^{(a)}$ ($a = 1, \ldots N^2 - 1$), of which $N - 1$ are Abelian generators $\Lambda^{(i)}$ ($i = 1, \ldots N - 1$).

The gauge transformed Abelian directions (Cartan generators) are denoted as

$$\hat{n}_i(x) = U(x)^\dagger \Lambda^{(i)} U(x). \tag{1}$$

Gluon fluctuations in the $\hat{n}_i$ directions are described by $c_\mu^{(i)}$, where μ is the Minkowski index. There is a covariant derivative which leaves the $\hat{n}_i$ invariant,

$$\hat{D}_\mu \hat{n}_i(x) \equiv (\partial_\mu + g\vec{V}_\mu(x)\times)\hat{n}_i(x) = 0, \tag{2}$$

where $\vec{V}_\mu(x)$ is of the form

$$\vec{V}_\mu(x) = c_\mu^{(i)}(x)\hat{n}_i(x) + \vec{C}_\mu(x), \quad \vec{C}_\mu(x) = g^{-1}\partial_\mu \hat{n}_i(x) \times \hat{n}_i(x). \tag{3}$$

The vector notation refers to the internal space, and summation is implied over $i = 1, \ldots N - 1$. For later convenience, we define

$$F_{\mu\nu}^{(i)}(x) = \partial_\mu c_\nu^{(i)}(x) - \partial_\nu c_\mu^{(i)}(x), \tag{4}$$

$$\vec{H}_{\mu\nu}(x) = \partial_\mu \vec{C}_\nu(x) - \partial_\nu \vec{C}_\mu(x) + g\vec{C}_\mu(x) \times \vec{C}_\nu(x) = \partial_\mu \hat{n}_i(x) \times \partial_\nu \hat{n}_i(x), \tag{5}$$

$$H_{\mu\nu}^{(i)}(x) = \vec{H}_{\mu\nu}(x) \cdot \hat{n}_i(x), \tag{6}$$

$$\vec{F}_{\mu\nu}^{(i)}(x) = F_{\mu\nu}^{(i)}(x)\hat{n}_i(x) + \vec{H}_{\mu\nu}(x). \tag{7}$$

The second last term in Equation (5) follows from the definition in Equation (3). Its being a cross-product is significant as it prevents μ, ν from having the same value. The Lagrangian contains the square of this value, namely

$$H_{\mu\nu}^{(i)}(x)H_{(i)}^{\mu\nu}(x) = \big(\partial_\mu \hat{n}_i(x) \times \partial_\nu \hat{n}_i(x)\big) \cdot (\partial^\mu \hat{n}_i(x) \times \partial^\nu \hat{n}_i(x)), \tag{8}$$

The form of Equation (3) might suggest the possibility of third or higher time derivatives in a quadratic Lagrangian, but we have now seen that the specific form of the Cho connection does not allow this.

The dynamical components of the gluon field in the off-diagonal directions of the internal space vectors are denoted by $\vec{X}_\mu(x)$, so if $\vec{A}_\mu(x)$ is the gluon field then

$$\vec{A}_\mu(x) = \vec{V}_\mu(x) + \vec{X}_\mu(x) = c_\mu^{(i)}(x)\hat{n}_i(x) + \vec{C}_\mu(x) + \vec{X}_\mu(x), \tag{9}$$

where

$$\vec{X}_\mu(x) \perp \hat{n}_i(x), \; \forall\, 1 \leq i < N\,, \quad \vec{D}_\mu = \partial_\mu + g\vec{A}_\mu(x). \tag{10}$$

The Lagrangian density is still

$$\mathcal{L}_{gauge}(x) = -\frac{1}{4}\vec{R}_{\mu\nu}(x) \cdot \vec{R}^{\mu\nu}(x), \tag{11}$$

where the field strength tensor of QCD expressed in terms of the CDG decomposition is

$$\vec{R}_{\mu\nu}(x) = \vec{F}_{\mu\nu}(x) + (\hat{D}_\mu \vec{X}_\nu(x) - \hat{D}_\nu \vec{X}_\mu(x)) + g\vec{X}_\mu(x) \times \vec{X}_\nu(x). \tag{12}$$

Gauge transformations are effected with a gauge parameter $\vec{\alpha}(x)$. Under a gauge transformation δ with $SU(2)$ parameter $\vec{\alpha}(x)$

$$\begin{aligned}
\delta\hat{V}(x) &= \hat{D}_\mu\vec{\alpha}(x) \\
\delta c_\mu(x) &= (\partial_\mu\vec{\alpha}(x)\cdot\hat{n}(x)), \\
\delta\hat{n}(x) &= \hat{n}(x)\times\vec{\alpha}(x), \\
\delta\vec{C}_\mu(x) &= (\partial_\mu\vec{\alpha}(x))_{\perp\hat{n}} + g\vec{C}_\mu(x)\times\vec{\alpha}(x), \\
\delta\vec{X}_\mu(x) &= g\,\vec{X}_\mu(x)\times\vec{\alpha}(x).
\end{aligned} \tag{13}$$

The form of the transform for $\vec{X}_\mu$ is the same as that for a coloured source, so that these components are sometimes described as "valence". This gauge transformation tell us two interesting things. The first is that the Abelian component c_μ combined with the Cho connection $\vec{C}_\mu$ is enough to represent the full Lorentz symmetry even without the valence components $\vec{X}_\mu$ Cho et al. [28,29] described as the "restricted" theory. The second is that the valence components transform like a source transforms. There is a corresponding situation in $N = 2$ Yang–Mills theory where the valence gluons are interpreted as colour sources. The importance of this observation is that we shall later discuss the possibility of mass generation for the valence gluons and this form for the gauge transformation leaves such mass terms covariant. We note however that a bare mass for $\vec{X}_\mu$ cannot be inserted artificially without spoiling renormalisability.

2.1.2. The Degrees of Freedom in the CDG Decomposition

Henceforth, we restrict ourselves to the $SU(2)$ theory, for which there is only one $\hat{n}$, and neglect the (i) indices.

The unit vector $\hat{n}$ posseses two DOFs and so its inclusion in the gluon field together with the Abelian component c_μ and the valence gluons $\vec{X}_\mu$ raises questions about the DOF of the decomposed gluon, with one paper [17] advocating the gauge condition

$$\hat{D}_\mu\vec{X}_\mu(x) = 0, \tag{14}$$

to remove two apparent extra degrees of freedom. The matter was sorted by Bae et al. [18], who demonstrated that the DOFs of $\hat{n}$ were not canonical but topological, indicating the embedding of the Abelian subgroup in the gauge group. The canonical DOFs are carried by the components c_μ, $\vec{X}_\mu$ and Equation (14) is a consistency condition expected of valence gluons. Kondo et al. [31] considered a stronger condition guaranteed not to be unaffected by Gribov copys.

The topological nature of $\hat{n}$ has significance beyond making the canonical DOFs add up correctly. As is well known, monopole configurations in gauge theories are topological configurations corresponding to the embedding of an Abelian subgroup. The other important consequence is that $\hat{n}$ does not have a canonical EOM from the Euler–Lagrange equation.

We took an alternative approach to this issue by applying a new method for finding the effects of degenerate variables called the Clairaut formalism. We further assumed that, as a unit vector, its dynamics were best described by angular variables.

2.2. CDG Decomposition of $SU(2)\times SU(2)$ in Euclidean Space

As is well known [10,13–16], the non-compact nature of the Lorentz group causes Lorentz gauge theories to be non-positive semi-definite. In fact, our attempts to apply the CDG decomposition to the Lorentz gauge field strength tensor in Minkowski space led to negative kinetic energy terms for some of the gauge fields (not shown). As demonstrated by Pak et al. [10,16], this can be avoided by Wick rotating the theory to Euclidean space and then either considering effective theories or finding a way to rotate back later without spoiling the quantum theory.

This procedure also rotates the internal Lorentz group to $SO(4)$ which is locally isomorphic to $SU(2)_R \times SU(2)_L$, corresponding to the right- and left-handed groups generated by

$$ {}_{\pm}\hat{e}_l \equiv \frac{1}{\sqrt{2}}(J_l \pm iK_l), \tag{15} $$

where J_l, K_l are the rotation and boost operators, respectively, and ${}_{\pm}\hat{e}_l$ is used to represent the corresponding direction in the internal space of the corresponding group. The two $SU(2)$ subgroups in our gauge theory, though separate, are not independent but are built from the same rotation and boost operators, albeit in combinations of opposite chirality. It follows that their respective Abelian directions must correspond, but represent operators of different chirality. We denote them $\hat{n}_R, \hat{n}_L$, respectively, using these suffices for other field objects also when appropriate, including ${}_R\hat{e}_l, {}_L\hat{e}_l$, and apply previously published analyses [1,3,6–8] to each symmetry group.

We apply the CDG decomposition to $SU(2)_R \times SU(2)_L$ gauge group. Their Abelian components we denote ${}_Rc_\mu$ and ${}_Lc_\mu$, respectively, and the valence components we denote as ${}_R\vec{X}_\mu$ and ${}_L\vec{X}_\mu$, respectively. For each chirality $\chi \in \{R, L\}$, we have the Cho connection

$$ {}_\chi C_\mu(x) = g^{-1}\partial_\mu\, {}_\chi\hat{n}(x) \times {}_\chi\hat{n}(x), \tag{16} $$

and monopole field strength

$$ \begin{aligned} {}_\chi\vec{H}_{\mu\nu} \equiv \partial_\mu {}_\chi\vec{C}_\nu(x) - \partial_\nu {}_\chi\vec{C}_\mu(x) + g\, {}_\chi\vec{C}_\mu(x) \times {}_\chi\vec{C}_\nu(x) = \partial_\mu\, \hat{n}_\chi(x) \times \partial_\nu\, \hat{n}_\chi(x) \\ \equiv {}_\chi H_{\mu\nu}(x)\, \hat{n}_\chi(x). \end{aligned} \tag{17} $$

Similarly defining field strengths ${}_\chi\vec{F}(x), {}_\chi\vec{R}_{\mu\nu}(x)$, we see from the direct product structure of the group that the Lagrangian is simply

$$ \mathcal{L}^E_{gauge}(x) = \frac{1}{4} \sum_{\chi \in \{R,L\}} {}_\chi\vec{R}_{\mu\nu}(x) \cdot {}_\chi\vec{R}_{\mu\nu}(x). \tag{18} $$

3. The Vacuum of $SU(2)_R \times SU(2)_L$

Since the component $SU(2)$ symmetry groups have generators mutually orthogonal in the internal space, their contributions to the ground state may be calculated independently and summed. Furthermore, their identical fundamental dynamics imply that ${}_\chi H_{\mu\nu}$ is independent of χ when we are not considering an internal vector and may be replaced with $H_{\mu\nu}$, which we do henceforth.

It is sufficient to calculate to one loop to find a non-zero monopole condensate in the effective action of $SU(2)$ Yang–Mills theory. The authors of [6–8] have shown this by a variety of methods. Useful material on this theory at one-loop order can also be found in references [32–34].

Calculating the relevant one-loop Feynman diagrams in Feynman gauge with dimensional regularisation [7,8], we have

$$ \Delta S_{eff} = -\frac{11g^2}{96} \sum_{\chi=R,L} \int d^4p\, {}_\chi\vec{F}_{\mu\nu}(p)\, {}_\chi\vec{F}_{\mu\nu}(-p) \left(\frac{2}{\epsilon} - \gamma - \ln\left(\frac{p^2}{\mu^2}\right)\right). \tag{19} $$

An imaginary part is generated by the $\ln \frac{p^2}{\mu^2}$ term only when the momentum p is timelike, leading to the well-known result [7,8,35] that it is the electric backgrounds are unstable but magnetic ones are not. Using this information, we then have the effective potential

$$ V = \frac{H^2}{g^2}\left[1 + \frac{11g^2}{24}\left(\ln \frac{\sqrt{H^2}}{\mu^2} - c\right)\right] \tag{20} $$

It should be remembered that this close parallel with the corresponding $N = 2$ calculation does not hold beyond one loop because then there are diagrams including fields from both $SU(2)$ subgroups.

Defining the running coupling $\bar{g}$ by [7,8]

$$\frac{\partial^2 V}{\partial H^2}\Big|_{\sqrt{H^2}=\bar{\mu}^2} = \frac{1}{\bar{g}^2}, \tag{21}$$

leads to a non-trivial local minimum at

$$\langle H \rangle = \bar{\mu}^2 \exp\Big(-\frac{24\pi^2}{11\bar{g}^2} + 1 \Big). \tag{22}$$

The specific value of H^2 is less important than knowing it has a strictly positive value lying in two orthogonal directions in the $SU(2)_R \times SU(2)_L$ internal space.

4. Application of Clairaut Formalism to the Rotation-Boost Decomposition of the Gravitational Connection

4.1. A Review of the Hamiltonian-Clairaut Formalism

Here, we review the main ideas and formulae of the Clairaut-type formalism for singular theories [26,36,37]. Let us consider a singular Lagrangian $L(q^A, v^A) = L^{\deg}(q^A, v^A)$, $A = 1, \ldots n$, which is a function of $2n$ variables (n generalised coordinates q^A and n velocities $v^A = \dot{q}^A = dq^A/dt$) on the configuration space $\mathsf{T}M$, where M is a smooth manifold, for which the Hessian's determinant is zero. Therefore, the rank of the Hessian matrix $W_{AB} = \frac{\partial^2 L(q^A, v^A)}{\partial v^B \partial v^C}$ is $r < n$, and we suppose that r is constant. We can rearrange the indices of W_{AB} in such a way that a non-singular minor of rank r appears in the upper left corner. Then, we represent the index A as follows: if $A = 1, \ldots, r$, we replace A with i (the "regular" or "canonical" index), and, if $A = r+1, \ldots, n$ we replace A with α (the "degenerate" or "non-canonical" index). Obviously, $\det W_{ij} \neq 0$, and $\operatorname{rank} W_{ij} = r$. Thus any set of variables labelled by a single index splits as a disjoint union of two subsets. We call those subsets regular (having Latin indices) and degenerate (having Greek indices). Canonical DOFs are obviously described by the former of these subsets while other DOFs can be placed in the second if their contribution to the Wronskian vanishes. As was shown in [26,36], the "physical" Hamiltonian can be presented in the form

$$H_{phys}\Big(q^A, p_i\Big) = \sum_{i=1}^{r} p_i V^i\Big(q^A, p_i, v^\alpha\Big) + \sum_{\alpha=r+1}^{n} B_\alpha\Big(q^A, p_i\Big) v^\alpha - L\Big(q^A, V^i\Big(q^A, p_i, v^\alpha\Big), v^\alpha\Big), \tag{23}$$

where the functions

$$B_\alpha\Big(q^A, p_i\Big) \stackrel{def}{=} \frac{\partial L(q^A, v^A)}{\partial v^\alpha}\Bigg|_{v^i = V^i(q^A, p_i, v^\alpha)} \tag{24}$$

are independent of the unresolved velocities v^α since $\operatorname{rank} W_{AB} = r$. Additionally, the r.h.s. of (23) does not depend on the degenerate velocities v^α

$$\frac{\partial H_{phys}}{\partial v^\alpha} = 0, \tag{25}$$

which justifies the term "physical". The Hamilton–Clairaut system which describes any singular Lagrangian classical system (satisfying the second-order Lagrange equations) has the form

$$\frac{dq^i}{dt} = \left\{q^i, H_{phys}\right\}_{phys} - \sum_{\beta=r+1}^{n} \left\{q^i, B_\beta\right\}_{phys} \frac{dq^\beta}{dt}, \quad i = 1, \ldots r \tag{26}$$

$$\frac{dp_i}{dt} = \left\{p_i, H_{phys}\right\}_{phys} - \sum_{\beta=r+1}^{n} \left\{p_i, B_\beta\right\}_{phys} \frac{dq^\beta}{dt}, \quad i = 1, \ldots r \tag{27}$$

$$\sum_{\beta=r+1}^{n} \left[\frac{\partial B_\beta}{\partial q^\alpha} - \frac{\partial B_\alpha}{\partial q^\beta} + \left\{B_\alpha, B_\beta\right\}_{phys}\right] \frac{dq^\beta}{dt} = \frac{\partial H_{phys}}{\partial q^\alpha} + \left\{B_\alpha, H_{phys}\right\}_{phys}, \qquad \alpha = r+1, \ldots, n \tag{28}$$

where the "physical" Poisson bracket (in regular variables q^i, p_i) is

$$\{X, Y\}_{phys} = \sum_{i=1}^{n-r} \left(\frac{\partial X}{\partial q^i}\frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial q^i}\frac{\partial X}{\partial p_i}\right). \tag{29}$$

Whether the variables $B_\alpha\left(q^A, p_i\right)$ have a non-trivial effect on the time evolution and commutation relations is equivalent to whether or not the so-called "q^α-field strength"

$$\mathcal{F}_{\alpha\beta} = \frac{\partial B_\beta}{\partial q^\alpha} - \frac{\partial B_\alpha}{\partial q^\beta} + \left\{B_\alpha, B_\beta\right\}_{phys} \tag{30}$$

is non-zero. The reader is referred to [26,27,36] for more details.

4.2. *The Contribution of the Clairaut Formalism*

4.2.1. q^α Curvature

Substituting in this notation, the angles ϕ, θ are seen, in parallel with our previously published analysis [25], to be degenerate DOFs with unresolved velocities. Indeed, their contribution to both Lagrangian and Hamiltonian vanishes when their derivatives vanish.

We use the CDG decomposition in which the embedding of a dominant direction $U(1)$ is denoted by $\hat{n}_\chi$ which, from the discussion in Section 2.2, is expressed by,

$$\hat{n}_\chi(x) \equiv \cos\theta(x)\sin\phi(x)\, {}_\chi\hat{e}_1 + \sin\theta(x)\sin\phi(x)\, {}_\chi\hat{e}_2 + \cos\phi(x)\, {}_\chi\hat{e}_3. \tag{31}$$

We note that the angles are ϕ, θ are independent of χ for the reasons discussed after Equation (15) and need not be labelled. The following will prove useful:

$$\begin{aligned}
\sin\phi(x)\, {}_\chi\hat{n}_\theta(x) &\equiv \int dy^4 \frac{d\hat{n}(x)}{d\theta(y)} = \sin\phi(x)\left(-\sin\theta(x)\, {}_\chi\hat{e}_1 + \cos\theta(x)\, {}_\chi\hat{e}_2\right), \\
{}_\chi\hat{n}_\phi(x) &\equiv \int dy^4 \frac{d\hat{n}_\chi(x)}{d\phi(y)} = \cos\theta(x)\cos\phi(x)\, {}_\chi\hat{e}_1 \\
&\quad + \sin\theta(x)\cos\phi(x)\, {}_\chi\hat{e}_2 - \sin\phi(x)\, {}_\chi\hat{e}_3.
\end{aligned} \tag{32}$$

For later convenience, we note that

$$\begin{aligned}
{}_\chi\hat{n}_{\phi\phi}(x) = -{}_\chi\hat{n}(x),\ {}_\chi\hat{n}_{\theta\theta}(x) &= -\sin\phi\, {}_\chi\hat{n}(x) - \cos\phi(x)\, {}_\chi\hat{n}_\phi(x)\,, \\
{}_\chi\hat{n}_{\theta\phi}(x) = 0,\ \ {}_\chi\hat{n}_{\phi\theta}(x) &= \cos\phi(x)\, {}_\chi\hat{n}_\theta(x),
\end{aligned} \tag{33}$$

and that the vectors ${}_\chi\hat{n} = {}_\chi\hat{n}_\phi \times {}_\chi\hat{n}_\theta$ form an orthonormal basis of the internal space. Substituting the above into the Cho connection in Equation (3) gives

$$
\begin{aligned}
g\,{}_\chi\vec{C}_\mu(x) &= (\cos\theta(x)\cos\phi(x)\sin\phi(x)\partial_\mu\theta(x) + \sin\theta(x)\partial\phi(x))\,{}_\chi\hat{e}_1 \\
&+ (\sin\theta(x)\cos\phi(x)\sin\phi(x)\partial_\mu\theta(x) - \cos\theta(x)\partial\phi(x))\,{}_\chi\hat{e}_2 - \sin^2\phi(x)\partial_\mu\theta(x)\,{}_\chi\hat{e}_3 \\
&= \sin\phi(x)\,\partial_\mu\theta(x)\,{}_\chi\hat{n}_\phi(x) - \partial_\mu\phi(x)\,{}_\chi\hat{n}_\theta(x)
\end{aligned} \tag{34}
$$

from which, it follows that

$$
g^2\,{}_\chi\vec{C}_\mu(x) \times {}_\chi\vec{C}_\nu(x) = \sin\phi(x)(\partial_\mu\phi(x)\partial_\nu\theta(x) - \partial_\nu\phi(x)\partial_\mu\theta(x))\hat{n}_\chi(x), \tag{35}
$$

where we again see that higher-order time derivatives are thwarted.

Since their Lagrangian terms do not fit the form of a canonical DOFs we consider them instead to be degenerate, having no canonical DOFs of their own but manifesting through their alteration of the EOMs of the dynamic variables. Finding these alterations first requires the Clairaut-related quantities

$$
\begin{aligned}
B_\phi(x) &= \int dy^3 \frac{\delta\mathcal{L}}{{}_x\partial_0\phi(x)} \\
&= \sum_{\chi=R,L} \int dy^3 \int dy_0\,\delta(x_0 - y_0)\Big(\sin\phi(y)_y\partial_\mu\theta(y)\hat{n}_\chi(y) \\
&\qquad + {}_\chi\hat{n}_\theta(y) \times {}_\chi\vec{X}_\mu(y)\Big) \cdot {}_\chi\vec{R}_{0\mu}(y)\,\delta^3(\vec{x} - \vec{y}) \\
&= \sum_{\chi=R,L} \Big(\sin\phi(x)\,\partial_\mu\theta(x)\hat{n}_\chi(x) + {}_\chi\hat{n}_\theta(x) \times {}_\chi\vec{X}_\mu(x)\Big) \cdot {}_\chi\vec{R}_{0\mu}(x),
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
B_\theta(x) &= \int dy^3 \frac{\delta\mathcal{L}}{{}_x\partial_0\theta(x)} \\
&= -\sum_{\chi=R,L} \int dy^3 \int dy_0\delta(x_0 - y_0)\sin\phi(y)\Big({}_y\partial_\mu\phi(y)\,\hat{n}_\chi(y) \\
&\qquad + \sin\phi(y)\,{}_\chi\hat{n}_\phi(y) \times {}_\chi\vec{X}_\mu(y)\Big) \cdot {}_\chi\vec{R}_{0\mu}(y)\,\delta^3(\vec{x} - \vec{y}) \\
&= -\sum_{\chi=R,L} \sin\phi(x)\Big(\partial_\mu\phi(x)\,\hat{n}_\chi(x) + {}_\chi\hat{n}_\phi(x) \times {}_\chi\vec{X}_\mu(x)\Big) \cdot {}_\chi\vec{R}_{0\mu}(x).
\end{aligned} \tag{37}
$$

$$
\frac{\delta B_\phi(x)}{\delta\theta(y)} = \sum_{\chi=R,L} \Big(\sin\phi(x)\,{}_\chi\hat{n}_{\theta\theta}(x) \times {}_\chi\vec{X}_\mu \cdot {}_\chi\vec{R}_{0\mu}(x) - {}_\chi T_\phi(x)\Big)\delta^4(x - y), \tag{38}
$$

$$
\begin{aligned}
\frac{\delta B_\theta(x)}{\delta\phi(y)} &= -\sum_{\chi=R,L} \Big(\cos\phi(x)\Big(\partial_\mu\phi(x)\,\hat{n}_\chi(x) + {}_\chi\hat{n}_\phi(x) \times {}_\chi\vec{X}_\mu(x)\Big) \cdot \Big({}_\chi\vec{R}_{0\mu}(x) + {}_\chi\vec{H}_{0\mu}(x)\Big) \\
&\qquad + {}_\chi T_\theta(x)\Big)\delta^4(x - y),
\end{aligned} \tag{39}
$$

where

$$
{}_\chi T_\phi(x) = \partial_k\Big[\sin\phi(x)\,\hat{n}_\chi \cdot {}_\chi\vec{R}_{0k}(x) - \Big(\sin\phi(x)\partial_k\theta(x) + {}_\chi\hat{n}_\theta(x) \times {}_\chi\vec{X}_k \cdot \hat{n}_\chi\Big)\partial_0\phi(x)\Big], \tag{40}
$$

$$
{}_\chi T_\theta(x) = -\partial_k\Big[\sin\phi(x)\Big(\hat{n}_\chi \cdot {}_\chi\vec{R}_{0k}(x) + \Big(\partial_k\phi(x) + {}_\chi\hat{n}_\phi(x) \times {}_\chi\vec{X}_k \cdot \hat{n}_\chi\Big)\partial_0\theta(x)\Big)\Big], \tag{41}
$$

are the surface terms arising from derivatives $\frac{\delta(\partial\theta)}{\delta\theta}$, $\frac{\delta(\partial\phi)}{\delta\phi}$ and the latin index k is used to indicate that only spacial indices are summed over.

This yields the q^α-curvature

$$\begin{aligned}\mathcal{F}_{\theta\phi}(x) &= \int dy^4 \Big(\frac{\delta B_\theta(x)}{\delta\phi(y)} - \frac{\delta B_\phi(x)}{\delta\theta(y)}\Big)\delta^4(x-y) + \{B_\phi(x), B_\theta(x)\}_{phys} \\ &= -\sum_{\chi=R,L} \cos\phi(x)\Big(\partial_\mu\phi(x)\,\hat{n}_\chi(x) + {}_\chi\hat{n}_\phi(x)\times{}_\chi\vec{X}_\mu(x)\Big)\cdot\Big({}_\chi\vec{R}_{0\mu}(x) + {}_\chi\vec{H}_{0\mu}(x)\Big) \\ &\quad - \sum_{\chi=R,L} \sin\phi(x)\,{}_\chi\hat{n}_{\theta\theta}(x)\times{}_\chi\vec{X}_\mu(x)\cdot{}_\chi\vec{R}_{\mu 0}(x) + \sum_{\chi=R,L}\Big({}_\chi T_\phi(x) - {}_\chi T_\theta(x)\Big).\end{aligned} \tag{42}$$

where we have used that the bracket $\{B_\phi, B_\theta\}_{phys}$ vanishes because B_ϕ and B_θ share the same dependence on the dynamic DOFs and their derivatives.

In earlier work on the Clairaut formalism [26,36], this was called the q^α-field strength, but we call it q^α-curvature in quantum field theory applications to avoid confusion.

This non-zero $\mathcal{F}^{\theta\phi}$ is necessary, and usually sufficient, to indicate a non-dynamic contribution to the conventional Euler–Lagrange EOMs. More significant is a corresponding alteration of the quantum commutators, with repurcussions for canonical quantisation and the particle number.

4.2.2. Corrections to the Equations of Motion

Generalising Equations (7.1,7.3,7.5) in [26] (see also the discussion around Equation (23)

$$\partial_0 q(x) = \{q(x), H_{phys}\}_{new} = \frac{\delta H_{phys}}{\delta p(x)} - \int dy^4 \sum_{\alpha=\phi,\theta} \frac{\delta B_\alpha(y)}{\delta p(x)}\partial_0\alpha(y), \tag{43}$$

the derivative of the Abelian component, complete with corrections from the monopole background is

$$\partial_0\, {}_\chi c_\sigma(x) = \frac{\delta H_{phys}}{\delta\, {}_\chi\Pi^\sigma(x)} - \int dy^4 \sum_{\alpha=\phi,\theta} \frac{\delta B_\alpha(y)}{\delta\, {}_\chi\Pi^\sigma(x)}\partial_0\alpha(y). \tag{44}$$

The effect of the second term is to remove the monopole contribution to $\frac{\delta H_{phys}}{\delta\, {}_\chi\Pi^\sigma}$. To see this, consider that, by construction, the monopole contribution to the Lagrangian and Hamiltonian is dependent on the time derivatives of θ, ϕ, so the monopole component of $\frac{\delta H_{phys}}{\delta\, {}_\chi\Pi^\sigma}$ is

$$\begin{aligned}\frac{\delta}{\delta\, {}_\chi\Pi_\sigma(x)} H_{phys}|_{\dot\theta\dot\phi} &= \frac{\delta}{\delta\, {}_\chi\Pi_\sigma(x)}\Big(\frac{\delta H_{phys}}{\delta\partial_0\theta(x)}\partial_0\theta(x) + \frac{\delta H_{phys}}{\delta\partial_0\phi(x)}\partial_0\phi(x)\Big) \\ &= \frac{\delta}{\delta\, {}_\chi\Pi_\sigma(x)}\Big(\frac{\delta L_{phys}}{\delta\partial_0\theta(x)}\partial_0\theta(x) + \frac{\delta L_{phys}}{\delta\partial_0\phi(x)}\partial_0\phi(x)\Big) \\ &= \frac{\delta}{\delta\, {}_\chi\Pi_\sigma(x)}\Big(B_\theta(x)\partial_0\theta(x) + B_\phi(x)\partial_0\phi(x)\Big),\end{aligned} \tag{45}$$

which is a consistency condition for Equation (44). This confirms the necessity of treating the monopole as a non-dynamic field.

We now observe that

$$\frac{\delta B_\theta(x)}{\delta\, {}_\chi c_\sigma(y)} = \frac{\delta B_\phi(x)}{\delta\, {}_\chi c_\sigma(y)} = 0, \tag{46}$$

from which it follows that the EOMs of ${}_\chi c_\sigma$ receives no correction. However, its $\{,\}_{phys}$ contribution, corresponding to the terms in the conventional EOM for the Abelian component, already contains a contribution from the monopole field strength.

Repeating the above steps for the valence gluons ${}_\chi\vec{X}_\mu$, assuming $\sigma \neq 0$ and combining

$$\hat{D}_0\,{}_\chi\vec{\Pi}_\sigma(x) = \frac{\delta H}{\delta\,{}_\chi\vec{X}_\sigma(x)} - \int dy^4 \sum_{\alpha=\phi,\theta} \frac{\delta B_\alpha(y)}{\delta\,{}_\chi\vec{X}_\sigma(x)}\partial_0\alpha(y). \tag{47}$$

with

$$\frac{\delta B_\phi(y)}{\delta\,{}_\chi\vec{X}_\sigma(x)} = -\Big(\Big(\sin\phi(y)_y\partial_\sigma\theta(y)\hat{n}_\chi(y) + {}_\chi\hat{n}_\theta(y) \times {}_\chi\vec{X}_\sigma(y)\Big) \times {}_\chi\vec{X}_0(y) \\ - {}_\chi\hat{n}_\phi(y)\hat{n}_\chi \cdot {}_\chi\vec{R}_{0\sigma}(y)\Big)\delta^4(x-y), \tag{48}$$

$$\frac{\delta B_\theta(y)}{\delta\,{}_\chi\vec{X}^\sigma(x)} = \Big(\Big(\partial_\sigma\phi(y)\hat{n}(y) + \sin\phi(y)\,\hat{n}_\phi(y) \times {}_\chi\vec{X}_\sigma(y)\Big) \times {}_\chi\vec{X}_0(y) \\ - \sin\phi(y)\,{}_\chi\hat{n}_\theta(y)\hat{n}_\chi(y) \cdot {}_\chi\vec{R}_{0\sigma}(y)\Big)\delta^4(x-y), \tag{49}$$

gives

$$\begin{aligned}\hat{D}_0\,{}_\chi\vec{\Pi}_\sigma(x) =& \frac{\delta H}{\delta\,{}_\chi\vec{X}_\sigma(x)} - \frac{1}{2}\Big(\Big(\sin\phi(x)(\partial_\sigma\phi(x)\partial_0\theta(x) - \partial_\sigma\theta(x)\partial_0\phi(x)\Big)\hat{n}_\chi(x) \\ &+ \Big(\sin\phi(x)\,{}_\chi\hat{n}_\phi(x)\partial_0\theta(x) - {}_\chi\hat{n}_\theta(x)\partial_0\phi(x)\Big) \times {}_\chi\vec{X}_\sigma(x)\Big) \times {}_\chi\vec{X}_0(x) \\ =& \frac{\delta H}{\delta\,{}_\chi\vec{X}_\sigma(x)} - \frac{1}{2}g^2\Big({}_\chi\vec{C}_\sigma(x) \times {}_\chi\vec{C}_0(x) + {}_\chi\vec{C}_0(x) \times {}_\chi\vec{X}_\sigma(x)\Big) \times {}_\chi\vec{X}_0(x).\end{aligned} \tag{50}$$

This is the converse situation of the Abelian gluon, where their derivatives ${}_\chi\vec{X}_\sigma$ is uncorrected while their EOM receives a correction which cancels the monopole's electric contribution to $\{\hat{D}_0\,{}_\chi\vec{X}_\sigma, H_{phys}\}_{phys}$. This is required by the conservation of topological current, but a further implication is that the monopole background, even if assumed to be present, does not contribute to the EOMs of motion and therefore makes no impact at the classical level. Note that this is strictly limited to the monopole field and the effects of backgrounds due to the dynamic fields are not affected. Monopole field contributions are not cancelled from quantum corrections however, although calculating loop effects is beyond the scope of this paper.

4.2.3. Corrections to the Commutation Relations

Corrections to the classical Poisson bracket correspond to corrections to the equal-time commutators in the quantum regime. We shall see corrections for commutators with fields of different $SU(2)_\chi$ representations even though there were no such crossover terms in the effective potential calculation.

Denoting conventional commutators as $[,]_{phys}$ and the corrected ones as $[,]_{new}$, for $\mu, \nu \neq 0$, we have

$$\begin{aligned}[{}_\chi c_\mu(x),\, {}_{\tilde{\chi}}c_\nu(z)]_{new} =& [\,{}_\chi c_\mu(x),\, {}_{\tilde{\chi}}c_\nu(z)]_{phys} \\ &- \int dy^4\Big(\frac{\delta B_\theta(y)}{\delta\,{}_\chi\Pi_\mu(x)}\mathcal{F}^{-1}_{\theta\phi}(z)\frac{\delta B_\phi(y)}{\delta\,{}_\chi\Pi_\nu(z)} - \frac{\delta B_\phi(y)}{\delta\,{}_{\tilde{\chi}}\Pi_\mu(x)}\mathcal{F}^{-1}_{\phi\theta}(z)\frac{\delta B_\theta(y)}{\delta\,{}_\chi\Pi_\nu(z)}\Big)\delta^4(x-z) \\ =& [\,{}_\chi c_\mu(x),\, {}_{\tilde{\chi}}c_\nu(z)]_{phys} \\ &- \sin\phi(x)\sin\phi(z)(\partial_\mu\phi(x)\partial_\nu\theta(z) - \partial_\nu\phi(z)\partial_\mu\theta(x))\mathcal{F}^{-1}_{\theta\phi}(z)\delta^4(x-z).\end{aligned} \tag{51}$$

The second term on the final line, after integration over d^4z, clearly becomes

$$H_{\mu\nu}(x)\sin\phi(x)\mathcal{F}^{-1}_{\theta\phi}(x), \tag{52}$$

indicating the role of the monopole condensate in the correction. By contrast, the commutation relations

$$[{}_\chi c_\mu(x),\, {}_{\tilde\chi}\Pi_\nu(z)]_{new} = [{}_\chi c_\mu(x),\, {}_{\tilde\chi}\Pi_\nu(z)]_{phys},$$
$$[{}_\chi \Pi_\mu(x),\, {}_{\tilde\chi}\Pi_\nu(z)]_{new} = [{}_\chi \Pi_\mu(x),\, {}_{\tilde\chi}\Pi_\nu(z)]_{phys}, \tag{53}$$

are unchanged. Nonetheless, the deviation from the canonical commutation shown in Equation (51) is inconsistent with the particle creation/annihilation operator formalism of conventional second quantisation, so that particle number is no longer well defined for the ${}_\chi c_\mu$ fields.

Repeating for the valence part,

$$[{}_\chi \Pi^a_\mu(x), {}_{\tilde\chi}\Pi^b_\nu(z)]_{new} \tag{54}$$
$$= [{}_\chi \Pi^a_\mu(x),\, {}_{\tilde\chi}\Pi^b_\nu(z)]_{phys} - \int dy^4 \Big(\frac{\delta B_\theta(y)}{\delta\, {}_\chi X^a_\mu(x)} \frac{\delta B_\phi(y)}{\delta\, {}_{\tilde\chi} X^b_\nu(z)} - \frac{\delta B_\phi(y)}{\delta X^a_\mu(x)} \frac{\delta B_\theta(y)}{\delta\, {}_{\tilde\chi} X^b_\nu(z)} \Big) \mathcal{F}^{-1}_{\theta\phi}(z)$$
$$= [{}_\chi \Pi^a_\mu(x),\, {}_{\tilde\chi} Pi^b_\nu(z)]_{phys} + \Big(\sin\phi(z) n^a_\phi(x) n^b_\theta(z)\, {}_\chi \vec{R}_{0\mu}(x)\cdot \hat{n}_\chi(x)\, {}_{\tilde\chi}\vec{R}_{0\nu}(z)\cdot\hat{n}_{\tilde\chi}(z)$$
$$- \sin\phi(x) n^a_\theta(x) n^b_\phi(z)\, {}_\chi \vec{R}_{0\mu}(z)\cdot\hat{n}_\chi(z)\, {}_{\tilde\chi}\vec{R}_{0\nu}(x)\cdot\hat{n}_{\tilde\chi}(x) \Big) \times \mathcal{F}^{-1}_{\theta\phi}(z)\, \delta^4(x-z), \tag{55}$$

where the second term on the final line, integrates over $d^4 z$ to become

$$(n^a_\phi(x) n^b_\theta(x) - n^a_\theta(x) n^b_\phi(x)) \sin\phi(x)\, {}_\chi \vec{R}^{0\mu}(x)\cdot\hat{n}_\chi(x)\, {}_{\tilde\chi}\vec{R}^{0\nu}(x)\cdot\hat{n}_{\tilde\chi}(x) \mathcal{F}^{-1}_{\theta\phi}(x), \tag{56}$$

while other relevant commutators are unchanged

$$[{}_\chi X^a_\mu(x),\, {}_{\tilde\chi}\Pi^b_\nu(z)]_{new} = [{}_\chi X^a_\mu(x),\, {}_{\tilde\chi}\Pi^b_\nu(z)]_{phys},$$
$$[{}_\chi X^a_\mu(x),\, {}_{\tilde\chi} X^b_\nu(z)]_{new} = [{}_\chi X^a_\mu(x),\, {}_{\tilde\chi} X^b_\nu(z)]_{phys}. \tag{57}$$

5. Effective Action

5.1. Particle Number and the Monopole Background

It is textbook knowledge that gravitational curvature spoils canonical quantisation, but our approach gives a detailed mechanism. It also provides some narrowly defined circumstances under which it may be salvaged. For monopole background ${}_\chi \vec{H}_{\mu\nu}$ the form of Equation (51) indicates that they would arise for ${}_\chi c_\sigma$ polarised along either of the μ, ν directions. The only way to avoid this is if ${}_\chi c_\sigma$ is polarised in the direction of the monopole field strength, requiring that the Abelian component of the connection propagate at a right angle to the monopole field strength. However, the form of the monopole field strength requires that a non-vanishing field must have a varying orientation in space, since it is proportional to the derivatives of the angles ϕ, θ. So even if the Abelian gauge component is propagating at a right angle to the monopole field strength with its polarisation in the direction of the field strength, in general this could not be assumed to continue as the orientation of the monopole field strength varied. However, if the variation were gradual over space in comparison to the wavelength of ${}_\chi c_\mu$, then it might continue to propagate while adjusting to the required orientations in a manner analogous to photon polarisation being rotated by successive, closely oriented, polarising filters. On the other hand, if the wavelength of ${}_\chi c_\mu$ is significant compared to the length scale of the field variation, then such a mechanism could not act and the particle's energy would be either absorbed or deflected by the condensate, effectively suppressing the longer wavelengths and providing a measure of the background curvature.

One important observation is that the background field is (Lorentz) magnetic, so that at any point in spacetime a reference frame exists where the monopole field and its associated potential lie entirely along the spatial directions.

The particle inconsistent contribution from Equation (54) only occurs in the presence of a background electric component of the monopole field strength, vanishing when the polarisation of ${}_\chi\vec{X}_\mu$ is orthogonal to the electric component of the background field. This restricts the polarisation for a transversally polarised field whose direction of propagation is not in the direction of this electric component, but not otherwise. Of course, the electric component of the background monopole field can always be removed by a suitable Lorentz transformation, but this still leaves the particle interpretation frame-dependent.

Some authors have argued that the valence gluons in two-colour QCD gain an effective mass term [20,21] via their quartic interaction with the non-trivial monopole condensate. A similar mechanism could apply to the valence components of this theory. Consider the following quartic term from Equations (11) and (12),

$$\begin{aligned}&\frac{g^2}{4}({}_\chi\vec{C}_\mu(x) \times_\chi \vec{X}_\nu(x)) \cdot ({}_\chi\vec{C}^\mu(x) \times {}_\chi\vec{X}^\nu(x)) \\ =&\frac{g^2}{4}({}_\chi\vec{C}_\mu(x) \cdot {}_\chi\vec{C}^\mu(x)\ {}_\chi\vec{X}_\nu(x) \cdot {}_\chi\vec{X}^\nu(x) - {}_\chi\vec{C}_\mu(x) \cdot {}_\chi\vec{X}^\mu(x)\ {}_\chi\vec{X}_\mu(x) \cdot {}_\chi\vec{C}^\mu(x)). \qquad (58)\end{aligned}$$

Remembering that the Lorentz monopole fields ${}_\chi\vec{C}_\mu$ have non-zero condensates yields the terms

$$\frac{g^2}{4}\langle {}_\chi\vec{C}_\mu(x) \cdot {}_\chi\vec{C}^\mu(x)\rangle\ {}_\chi\vec{X}_\nu(x) \cdot {}_\chi\vec{X}^\nu(x), \qquad (59)$$

so that the monopole condensate is seen to generate a mass term for the valence component. Such a mass term is covariant under the gauge transformation because, as shown in the discussion of Equation (13), the valence components transform as sources although explicitly adding a mass term for these fields would spoil renormalisability. In this case the valence components could also be longitudinally polarised. With longitudinal polarisation the only restriction is that the direction of propagation be orthogonal to the background electric component of the monopole field strength. The valence component might therefore enjoy a limited particle interpretation under a range of circumstances.

We observe that the two monopole field strengths ${}_R\vec{H}_{\mu\nu}$, ${}_L\vec{H}_{\mu\nu}$ sum to give a net field strength lying purely along the rotation directions in the internal space. Exactly how this affects the observed dynamics of the theory, or even if it does, is unclear. We were unable to find a linear combination of the gauge fields to separate rotation and boost generators which was equivalent to the original theory. If there is an effect, then a reasonable scenario is that the coupling to linear momentum would dominate that to rotational momentum at large distances, as determined by the length scale of the condensate.

5.2. The Hilbert–Einstein Term

Kim and Pak [10] considered the effects of a torsion condensate. They found the resulting background field strength, if constant, spontaneously generated an EH term if the curvature tensor is expanded around it (see the discussion of Equation (45) in their paper [10]). EH terms have been shown to stabilise theories with higher-order derivatives by rendering the propagator poles gauge invariant [38,39] and Kim and Pak suggest that this may stabilise their theory also. Since our background is attributable to an Abelian background field, we expect the effective theory to have an Abelianised EH term, similar to that derived by Cho et al. [28,29] when applying the CDG decomposition to the Levi-Civita tensor. Such details must await further work, but we are encouraged to believe that the theory may be Wick rotated back to Lorentz space for a positive semi-definite effective theory. Not only do all quantum fields have kinetic terms with the correct sign, but the Lagrangian's lowest-order derivative terms come from an emergent term sometimes added to rectify the non-semi-positive definiteness.

6. Discussion

We have applied the CDG decomposition to a Lorentz gauge theory and confirmed that it has a monopole condensate at one loop. Using the Clairaut formalism, we have found how the monopole background modifies the canonical EOMs for the physical DOFs. Lorentz gauge theory has the problem of being non-positive semi-definite, which can be handled by adding a EH term. We did not add such a term but instead postponed the problem by Wick rotating the theory into Euclidean space, where the Lorentz gauge group becomes locally isomorphic to $SU(2)_R \times SU(2)_L$. We found the spontaneous generation of a vacuum condensate which others have argued [10,16] leads to an effective Hilbert–Einstein term.

The CDG decomposition introduces an internal unit vector to indicate the local internal direction of the Abelian subgroup of the gauged symmetry group. However, the unit vector used to specify this subgroup does not form a canonical EOM and is degenerate. If we expand it in terms of its angular dependence, since its information content is purely directional, then those angles are also degenerate and we do not derive canonical EOMs for them. They do however add additional terms with important consequences for the theory's physics. They may not be ignored therefore, but require appropriate theoretical tools to analyse them. The authors addressed these issues in a previous analysis of QCD. The purpose of this paper was to do so for a theory relevant to gravity. The main advantages of working in a gauged Lorentz theory for us is that the gauge fields have quadratic kinetic terms well suited to our Clairaut-based approach in addition to the opportunity to apply analyses and even results from $SU(2)$ Yang–Mills theories.

We have not considered the effects of matter fields in the fundamental representation. We do note in passing that differences in this part of the spectrum must lead to variations in the magnitude for the monopole condensate, so the differences in their matter spectra suggest that this theory has significantly different infrared behaviour from that of $SU(2)$ QCD.

We also observe that the net monopole condensate lies in a direction of a rotation generator. We have not been able to derive corresponding canonical DOFs to reflect this, so the physical significance of this observation, if any, remains obscure.

We have left the inclusion of translation symmetry to subsequent work. A full gravitational theory must of course include the full Poincaré symmetry group, but we submit that our Lorentz-only theory makes a sufficiently good approximation to indicate some relevant phenomenology.

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References

1. Savvidy, G.K. Infrared instability of the vacuum state of gauge theories and asymptotic freedom. *Phys. Lett.* **1977**, *B71*, 133. [CrossRef]
2. Nielsen, N.K.; Olesen, P. An unstable Yang-Mills field mode. *Nucl. Phys.* **1978**, *B144*, 376. [CrossRef]
3. 't Hooft, G. Topology of the Gauge Condition and New Confinement Phases in Nonabelian Gauge Theories. *Nucl. Phys.* **1981**, *B190*, 455. [CrossRef]
4. Cho, Y.M. Colored monopoles. *Phys. Rev. Lett.* **1980**, *44*, 1115–1118. [CrossRef]
5. Duan, Y.S.; Ge, M.L. $SU(2)$ gauge theory and electrodynamics of N moving magnetic monopoles. *Sci. Sinica* **1979**, *11*, 1072–1075.
6. Cho, Y.M.; Pak, D.G. Monopole condensation in $SU(2)$ QCD. *Phys. Rev.* **2002**, *D65*, 074027.
7. Cho, Y.M.; Walker, M.L. Stability of monopole condensation in $SU(2)$ QCD. *Mod. Phys. Lett.* **2004**, *A19*, 2707–2716. [CrossRef]
8. Cho, Y.M.; Walker, M.L.; Pak, D.G. Monopole condensation and confinement of color in $SU(2)$ QCD. *JHEP* **2004**, *05*, 073. [CrossRef]
9. Kay, D.; Kumar, A.; Parthasarathy, R. Savvidy vacuum in $SU(2)$ Yang-Mills theory. *Mod. Phys. Lett.* **2005**, *A20*, 1655–1662. [CrossRef]
10. Kim, S.W.; Pak, D.G. Torsion as a dynamic degree of freedom of quantum gravity. *Class. Quantum Gravity* **2008**, *25*, 065011. [CrossRef]

11. Pak, D. Confinement, vacuum structure: from QCD to quantum gravity. *Nucl. Phys. A* **2010**, *844*, 115c–119c. [CrossRef]
12. Cho, Y.M.; Pak, D.G.; Park, B.S. A Minimal model of Lorentz gauge gravity with dynamical torsion. *Int. J. Mod. Phys. A* **2010**, *25*, 2867–2882. [CrossRef]
13. Utiyama, R. Invariant theoretical interpretation of interaction. *Phys. Rev.* **1956**, *101*, 1597. [CrossRef]
14. Kibble, T. Lorentz invariance and the gravitational field. *J. Math. Phys.* **1961**, *2*, 212–221. [CrossRef]
15. Sciama, D. The physical structure of general relativity. *Rev. Mod. Phys.* **1964**, *36*, 463. [CrossRef]
16. Pak, D.; Kim, Y.; Tsukioka, T. Lorentz gauge theory as a model of emergent gravity. *Phys. Rev. D* **2012**, *85*, 084006. [CrossRef]
17. Shabanov, S.V. Yang–Mills theory as an Abelian theory without gauge fixing. *Phys. Lett.* **1999**, *B463*, 263–272. [CrossRef]
18. Bae, W.S.; Cho, Y.M.; Kimm, S.W. Qcd versus skyrme-faddeev theory. *Phys. Rev.* **2002**, *D65*, 025005.
19. Shabanov, S.V. An effective action for monopoles and knot solitons in Yang-Mills theory. *Phys. Lett.* **1999**, *B458*, 322–330. [CrossRef]
20. Kondo, K.I.; Murakami, T.; Shinohara, T. BRST symmetry of $SU(2)$ Yang-Mills theory in Cho-Faddeev-Niemi decomposition. *Eur. Phys. J.* **2005**, *C42*, 475–481. [CrossRef]
21. Kondo, K.I. Gauge-invariant gluon mass, infrared Abelian dominance and stability of magnetic vacuum. *Phys. Rev.* **2006**, *D74*, 125003. [CrossRef]
22. Lavrov, P.M.; Merzlikin, B.S. Legendre transformations and Clairaut-type equations. *Phys. Lett.* **2016**, *B756*, 188–193. [CrossRef]
23. Ren, J.; Wang, H.; Wang, Z.; Qu, F. The Wu-Yang potential of Magnetic Skyrmion from $SU(2)$ Flat Connection. *Sci. China Phys. Mech. Astron.* **2019**, *62*, 950021, [CrossRef]
24. Cho, Y.; Hong, S.T.; Kim, J.; Park, Y.J. Dirac quantization of restricted QCD. *Mod. Phys. Lett. A* **2007**, *22*, 2799–2813. [CrossRef]
25. Walker, M.L.; Duplij, S. Cho-Duan-Ge decomposition of QCD in the constraintless Clairaut-type formalism. *Phys. Rev. D* **2015**, *91*, 064022. [CrossRef]
26. Duplij, S. Generalized duality, Hamiltonian formalism and new brackets. *J. Math. Phys. Anal. Geom.* **2014**, *10*, 189–220. [CrossRef]
27. Duplij, S. Formulation of singular theories in a partial Hamiltonian formalism using a new bracket and multi-time dynamics. *Int. J. Geom. Methods Mod. Phys.* **2015**, *12*, 1550001. [CrossRef]
28. Cho, Y.; Oh, S.; Kim, S. Abelian dominance in Einstein's theory. *Class. Quantum Gravity* **2012**, *29*, 205007. [CrossRef]
29. Cho, Y.M.; Oh, S.H.; Park, B.S. Abelian decomposition of Einstein's theory: Restricted gravity. *Grav. Cosm.* **2015**, *21*, 257–269. [CrossRef]
30. Schanbacher, V. Gluon propagator and effective lagrangian in QCD. *Phys. Rev.* **1982**, *D26*, 489. [CrossRef]
31. Kondo, K.I.; Murakami, T.; Shinohara, T. Yang–Mills theory constructed from Cho-Faddeev-Niemi decomposition. *Prog. Theor. Phys.* **2006**, *115*, 201–216. [CrossRef]
32. Itzykson, C.; Zuber, J.B. *Quantum Field Theory*; McGraw-Hill: New York, NY, USA, 2012.
33. Peskin, M.; Schroeder, D. *An Introduction to Quantum Field Theory*; Addison-Wesley: Reading, MA, USA, 1995.
34. Weinberg, S. *The Quantum Theory of Fields*; Cambridge University Press: Cambridge, UK, 1996; Volumes 1–3.
35. Schwinger, J. On gauge invariance and vacuum polarization. *Phys. Rev.* **1951**, *82*, 664. [CrossRef]
36. Duplij, S. A new Hamiltonian formalism for singular Lagrangian theories. *J. Kharkov Univ. Ser. Nuclei Part. Fields* **2011**, *969*, 34–39.
37. Duplij, S. Constraintless Hamiltonian formalism and new brackets. In *Exotic Algebraic and Geometric Structures in Theoretical Physics*; Duplij, S., Ed.; Nova Publishers: New York, NY, USA, 2018; pp. 197–230.
38. Antoniadis, I.; Tomboulis, E. Gauge invariance and unitarity in higher-derivative quantum gravity. *Phys. Rev. D* **1986**, *33*, 2756. [CrossRef] [PubMed]
39. Johnston, D. Sedentary ghost poles in higher derivative gravity. *Nucl. Phys. B* **1988**, *297*, 721–732. [CrossRef]